A Mathematical Analysis of the Division Rules of Cities for Political Redistricting

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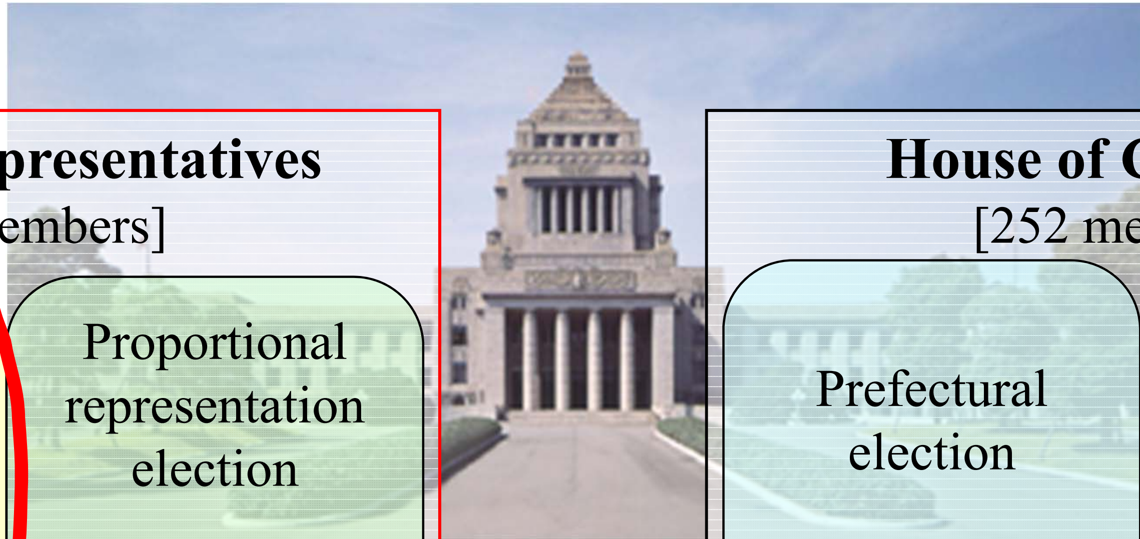
Overview

1. The Election System in Japan
   - How Diet members were elected
2. Mathematical Approach for the Redistricting Problem
3. The Exceptional Divide Rules in Japan
4. Some Results & Proposals
5. Conclusions & Future Works

The Election System in Japan



the Diet

the Lower House

House of Representatives
[480 members]

Single-seat constituency election	Proportional representation election
300 members	180 members

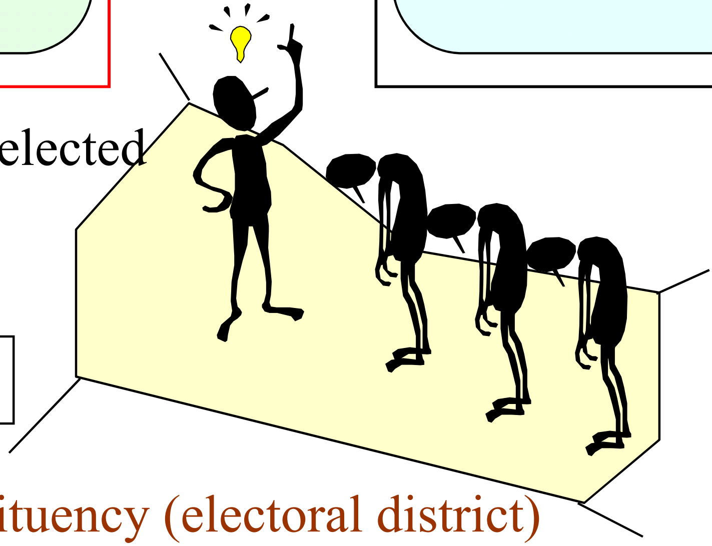
the Upper House

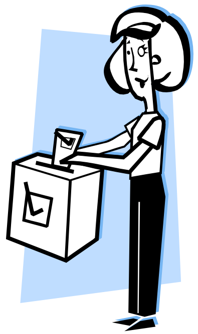
House of Councilors
[252 members]

Prefectural election	Proportional representation election
152 members	100 members

One candidate is elected

Single-seat constituency system



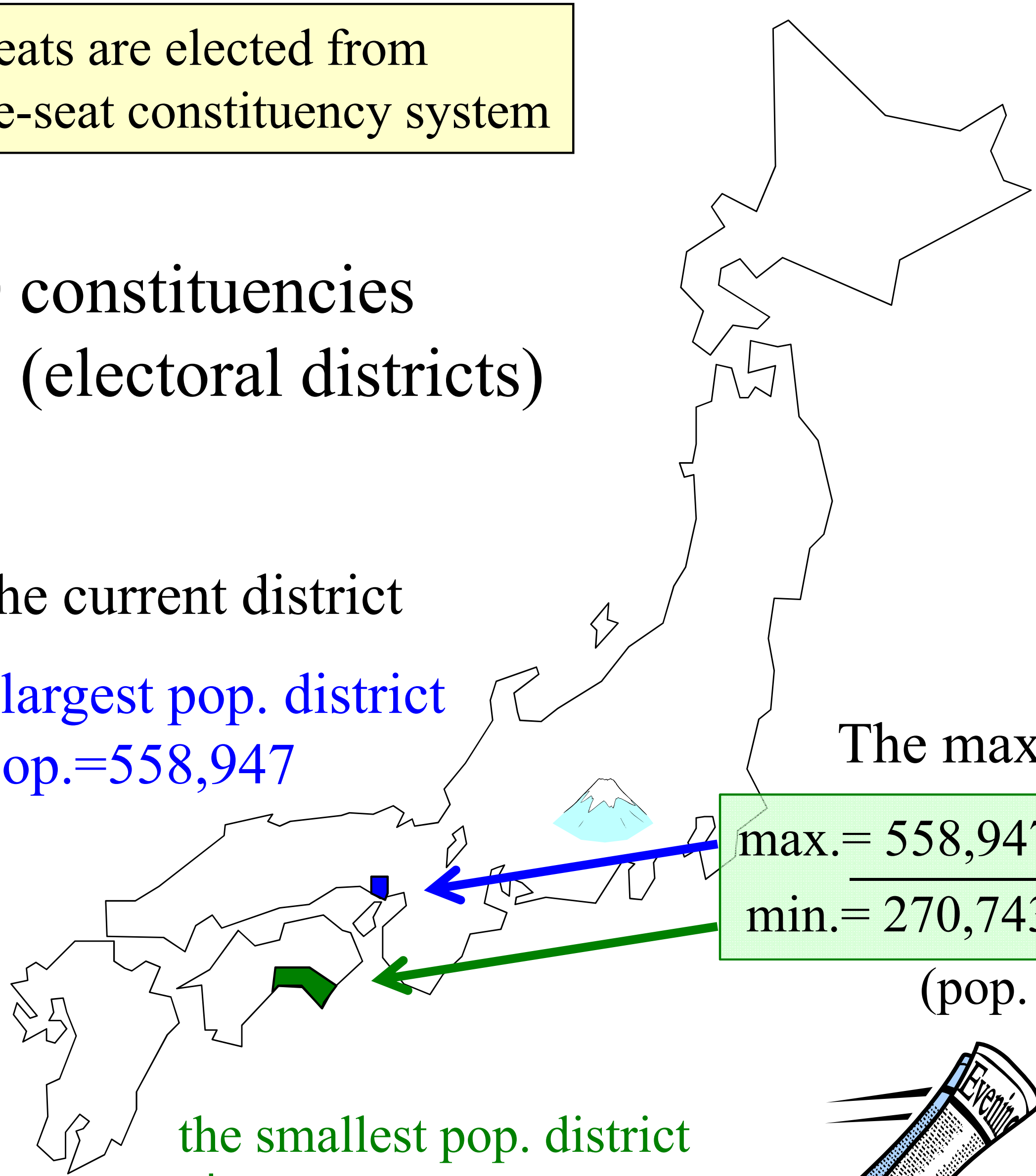


One constituency (electoral district)

The maximum population disparity

300 seats are elected from
Single-seat constituency system

300 constituencies
(electoral districts)



Districting rules

① Contiguous
② Do not divide a city
③ Disparity ratio < 2

One man, one vote

Constitutional principle

In the current district

the largest pop. district
＃pop.=558,947

The maximum pop. disparity ratio

$$\frac{\text{max.}= 558{,}947}{\text{min.}= 270{,}743} = \mathbf{2.064}$$

(pop. census 2000)

$$\frac{569{,}829}{258{,}687} = \mathbf{2.203}$$

(pop. census 2005)

the smallest pop. district
＃pop.=270,743

Objectionable!
Political problem

Districts planning process

1st phase

Apportionment

to 47 prefectures

253

47

300 seats

apportioned to 47 pref.
by the Largest Remainders Method (LRM)

Main cause!

does not match proportionality unconditionally

+1 seat rule

2nd phase

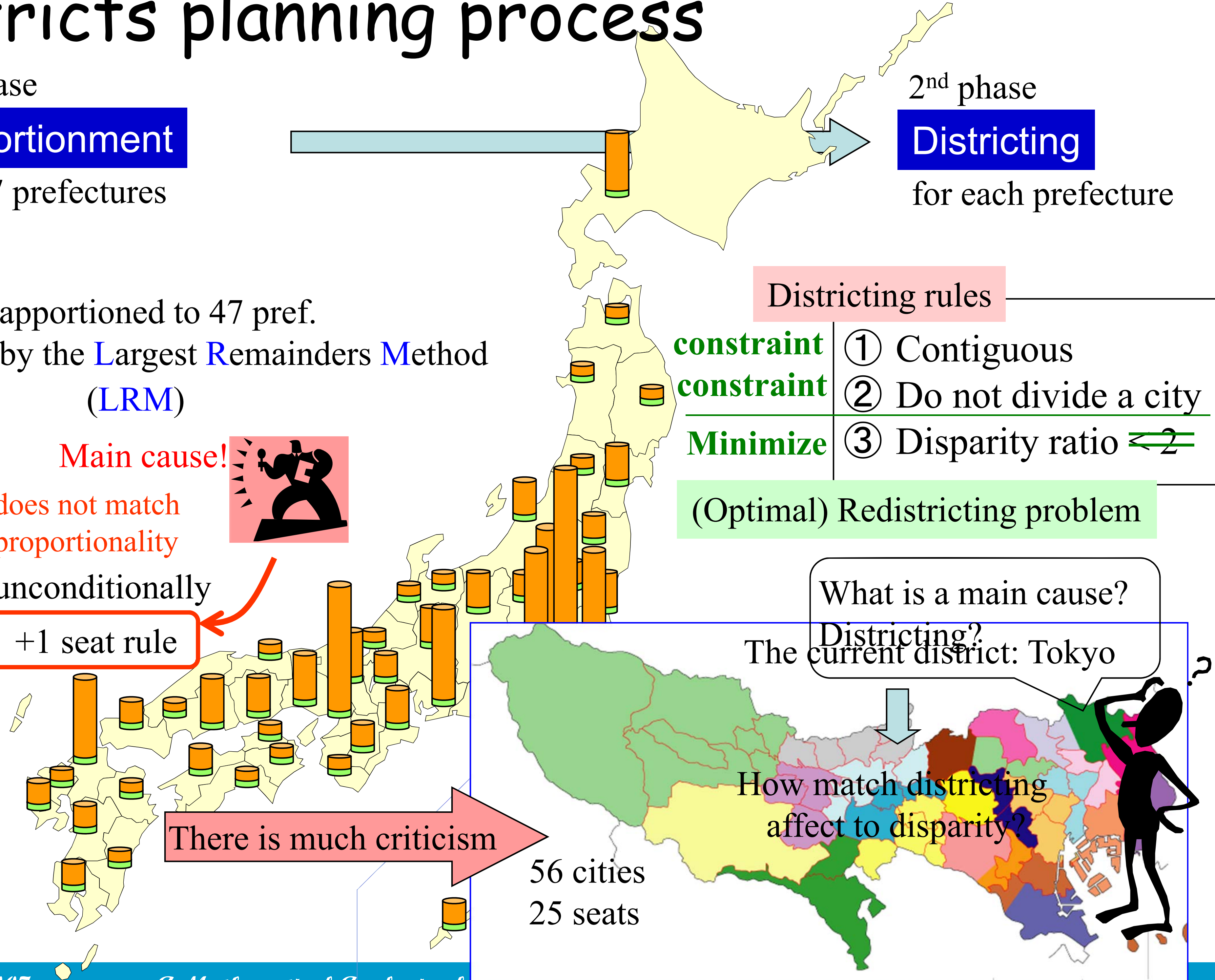
Districting

for each prefecture

Districting rules	
constraint	① Contiguous
constraint	② Do not divide a city
Minimize	③ Disparity ratio ~~≤ 2~~

(Optimal) Redistricting problem

What is a main cause? Districting?



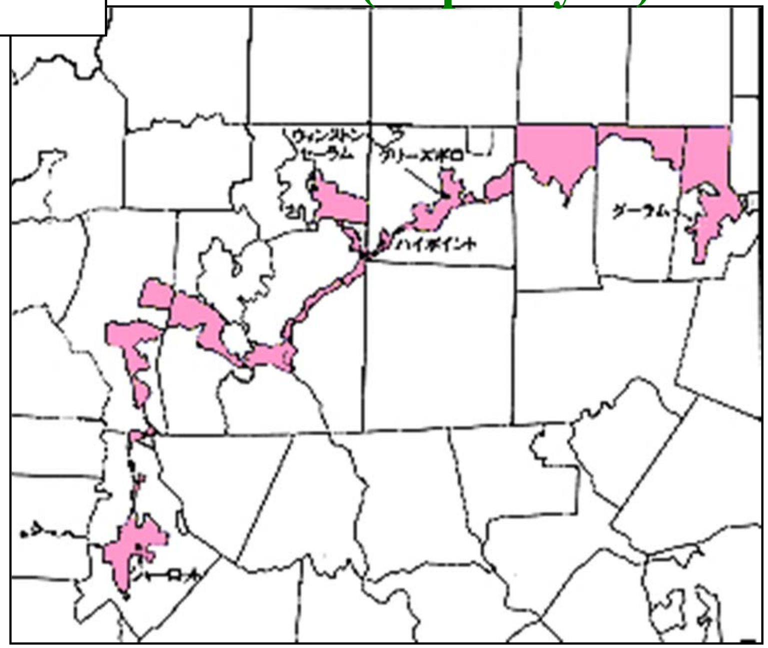
(Optimal) Redistricting Problem

- Previous works in U.S.
  - Mehrotra,Johnson,Nemhauser(1998) obtained the optimal district(46cities,6seats) by column generation technique.

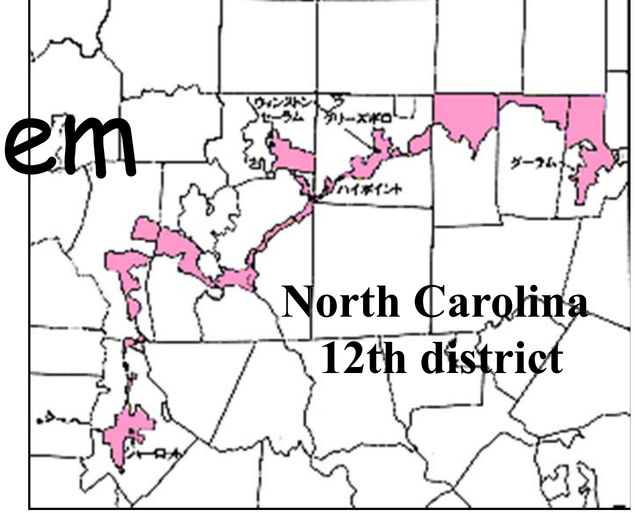
Japan	Redistricting problem	U.S.
constraint	① Contiguous	constraint
constraint	② Do not divide a city	ignore ⇄ object Compactness
Minimize	③ Disparity ratio ~~≤ 2~~	constraint (disparity=1)

North Carolina 12th district

gerrymander?



(Optimal) Redistricting Problem



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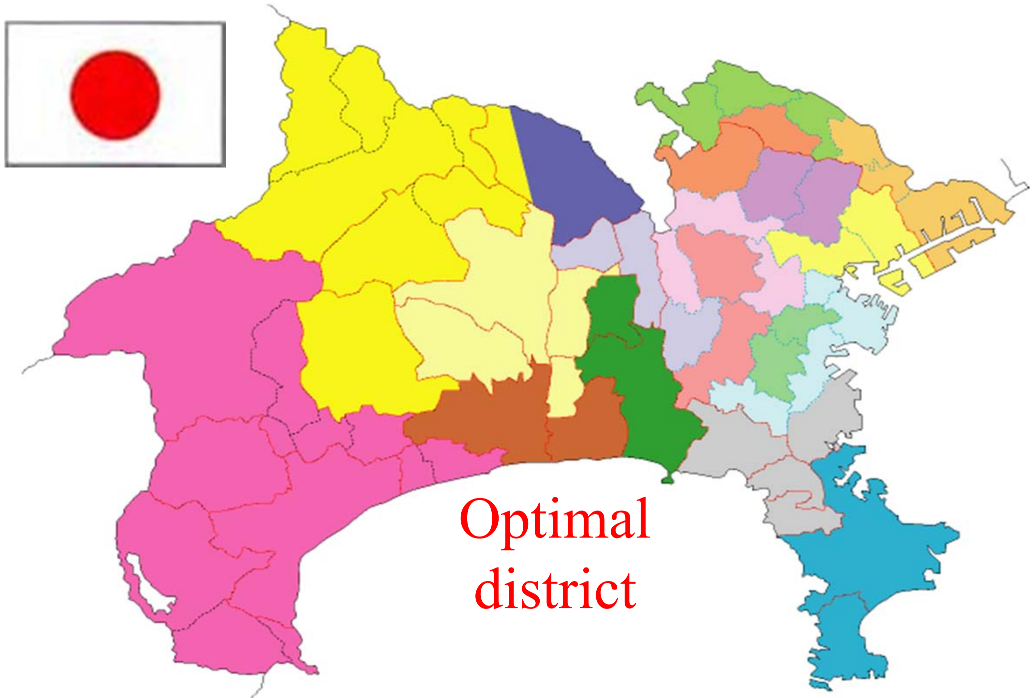
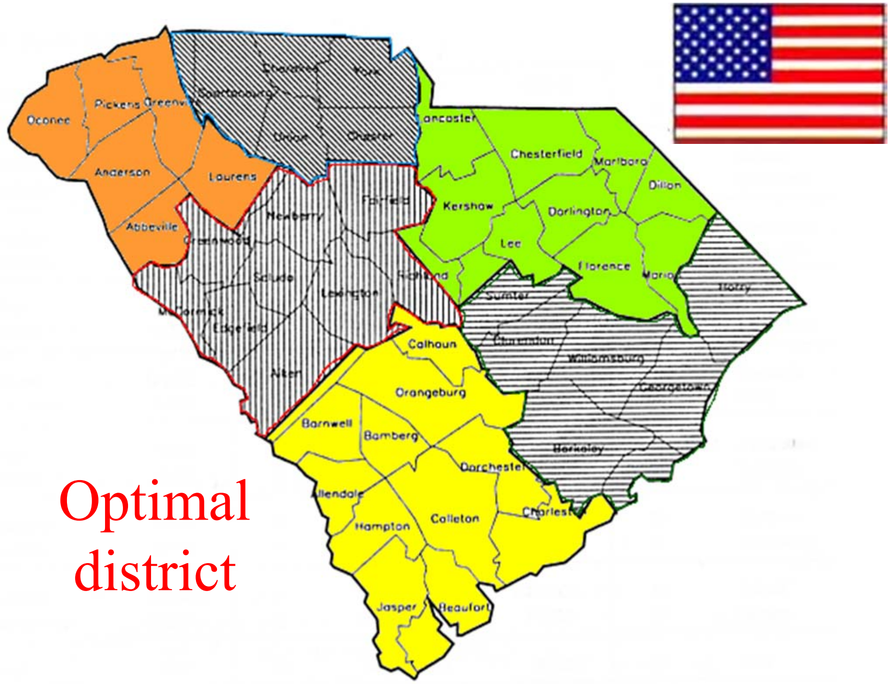


Kanagawa (city:49,district:18)
(Nemoto & Hotta 2002)



South Carolina(city:46,district:6)
(Mehrotra, Johnson&Nemhauser 1998)

(Optimal) Redistricting Problem

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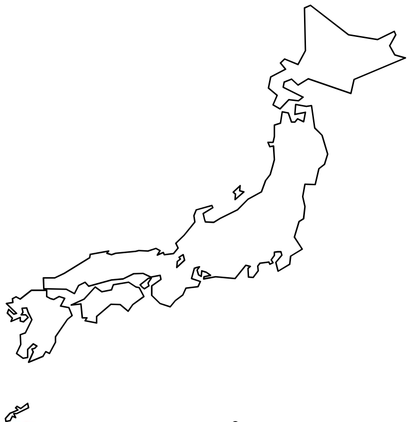


Japan	Redistricting problem	U.S.
constraint	① Contiguous	constraint
constraint	② Do not divide a city	ignore ⇄ object Compactness
Minimize	③ Disparity ratio ~~≤ 2~~	constraint (disparity=1)

- Previous works in Japan
  - Sakaguchi-Wada(2000) found opt.sol. (11 pref., ≦5 seats) by B.-and-B.

Tokyo

56 citeis
25 seats

Osaka

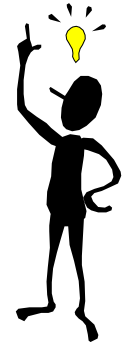
65 cities
19 seats

Approach

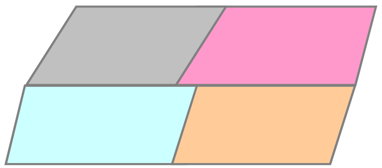
- Modeling

INPUT 300 seats, pop. of 47 pref. → **Apportionment Prob.** by **1+LRM** → OUTPUT 47 apportioned seats

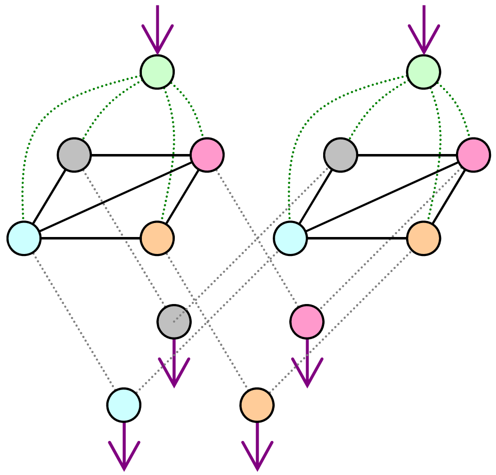
INPUT + pop. of cities → **47 Redistricting Prob.** → OUTPUT 47 redistricting plan



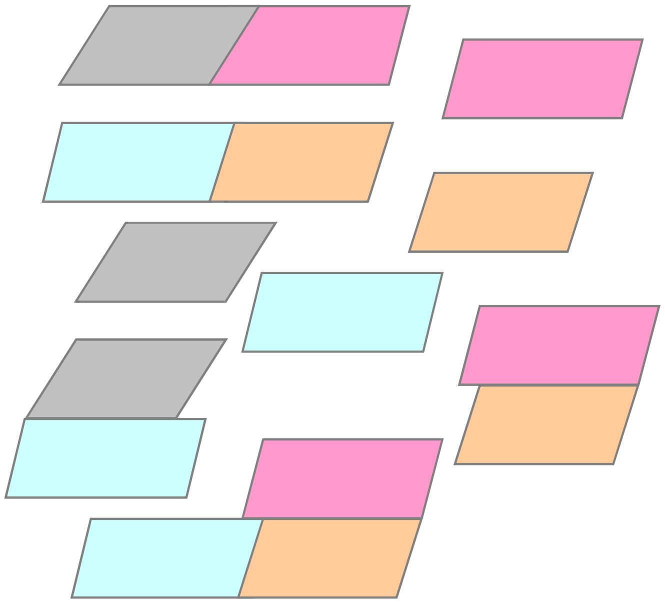
ex) 4 cities → 2 districts



graph partition type



set partition type



0-1 IP modeled by both the set partition type and the graph partition type

Formulation

- set partition type

Given appropriate subsets of cities, select k subsets partitioned pref.

$$
\begin{aligned}
\text{min.}\quad & u/l \\
\text{s.t.}\quad & q_j x_j \le u \quad (j=1,\ldots,|\beta|) \\
& \alpha(1-x_j)+q_j x_j \ge l \quad (j=1,\ldots,|\beta|) \\
& \sum_{j=1,\ldots,|\beta|} b_{ij} x_j = 1 \quad (i \in N) \\
& \sum_{j=1,\ldots,|\beta|} x_j = m \\
& x_j \in \{0,1\} \quad (j=1,\ldots,|\beta|)
\end{aligned}
$$

- graph partition type

Given city adjacency graph, divide into k connected subgraphs

$$
\begin{aligned}
\text{min.}\quad & u/l \\
\text{s.t.}\quad & l \le \sum_{i\in N} p_i z_{ik} \le u \quad (k \in M) \\
& \sum_{a\in\delta^- v_i^k} f(a) = \sum_{a\in\delta^+ v_i^k} f(a) \quad (i \in N, k \in M) \\
& f(a) \ge 0 \quad (a \in \bar{A}) \\
& f\left((s^k, v_i^k)\right) = \beta y_{ik} \quad (i \in N, k \in M) \\
& \sum_{i\in N} y_{ik} = 1 \quad (k \in M) \\
& y_{ik} \in \{0,1\} \quad (i \in N, k \in M) \\
& \sum_{a\in\delta^- v_i^k} f(a) = \beta z_{ik} \quad (i \in N, k \in M) \\
& z_{ik} \le f\left((v_i^k, t_i)\right) \quad (i \in N, k \in M) \\
& \sum_{k\in M} z_{ik} = 1 \quad (i \in N) \\
& z_{ik} \in \{0,1\} \quad (i \in N, k \in M)
\end{aligned}
$$

Approach & Results

Results

INPUT: 300 seats, pop. of 47 pref. → **Apportionment Prob.** (by **1+LRM**) → OUTPUT: 47 apportioned seats

INPUT: + pop. of cities → **47 Redistricting Prob.** → OUTPUT: 47 redistricting plan

↓

47 optimal sol.
(the optimal districts plan)

↓

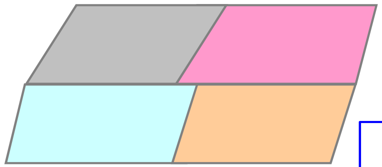
limit!

the limit of reduction
in the disparity ratio is **1.977**
(population census **2000**)

↓

the limit of reduction
in the disparity ratio is **2.153**
(population census **2005**)

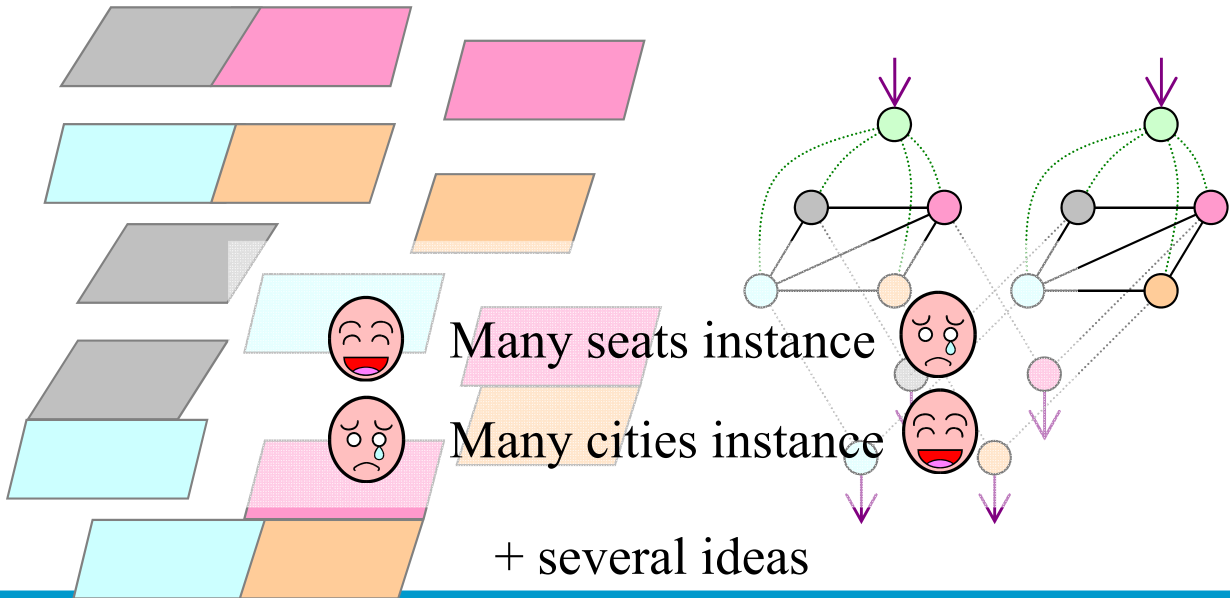
ex) 4 cities → 2 districts



0-1 IP modeled

set partition type

graph partition type



Many seats instance

Many cities instance

+ several ideas

Solved by
CPLEX9.0 &
OPL Studio 3.7

Results (2006)

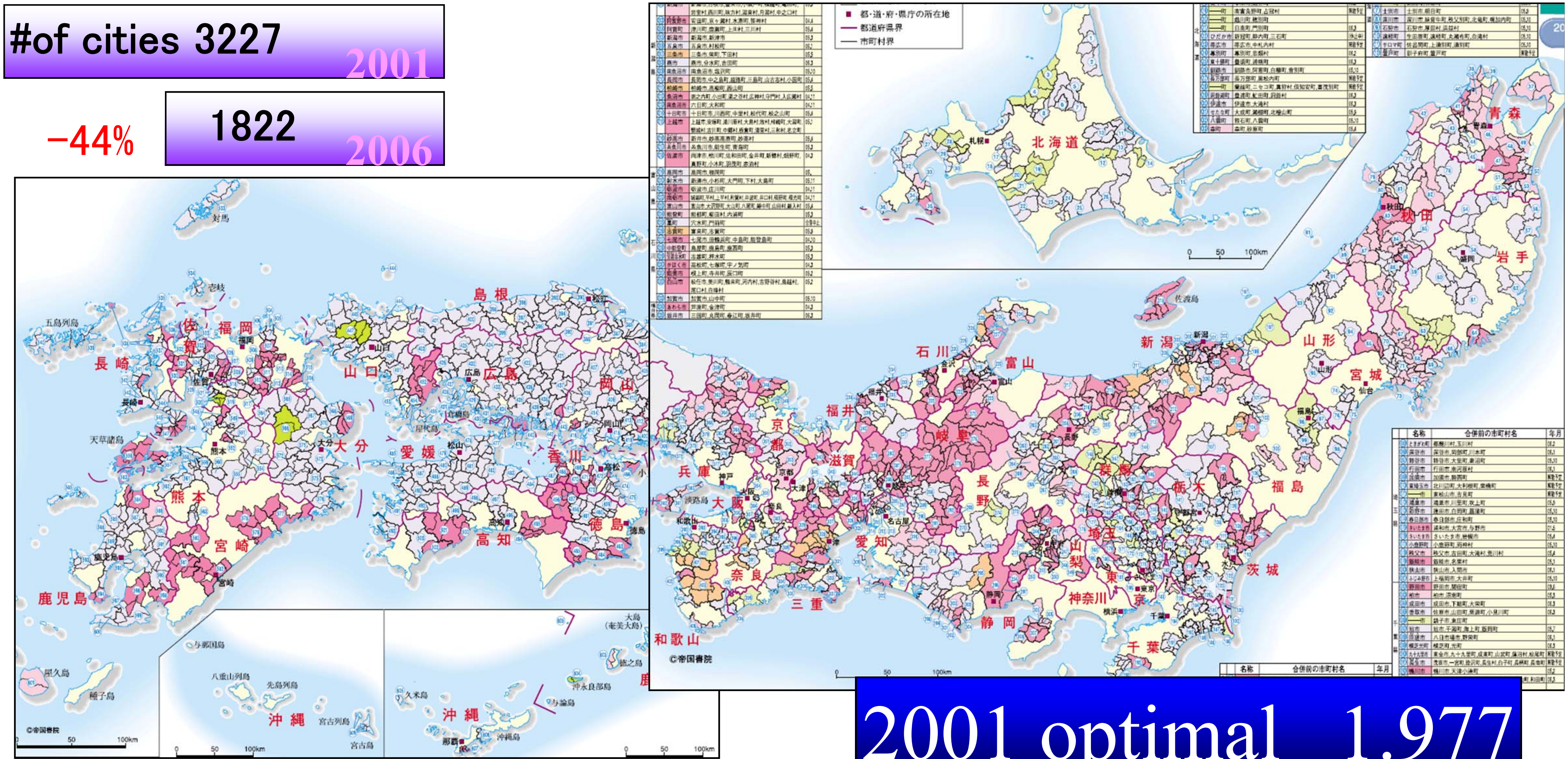
In Japan, the structural change has arisen from the municipal merger assistance plan

- Research the effect of the Great Municipal Merger in Heisei Era

#of cities 3227 2001

−44%

1822 2006



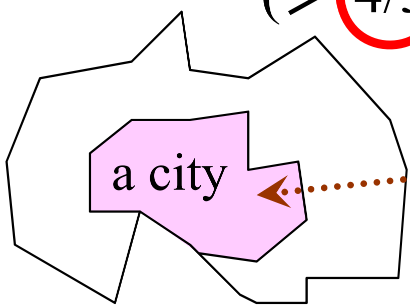
2001 optimal 1.977

2006 optimal 2.153

Current divide Rule

ex) pop.=600,000
app. seats=2

- What is a main cause of the disparity ?
  - divide rule
    - population of a city is **too big** (> 4/3 × average)

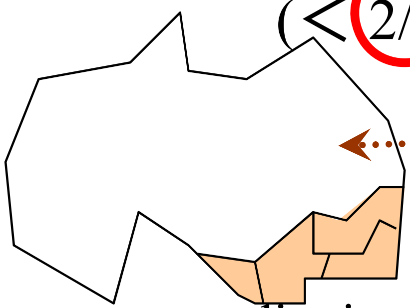


must divide the city

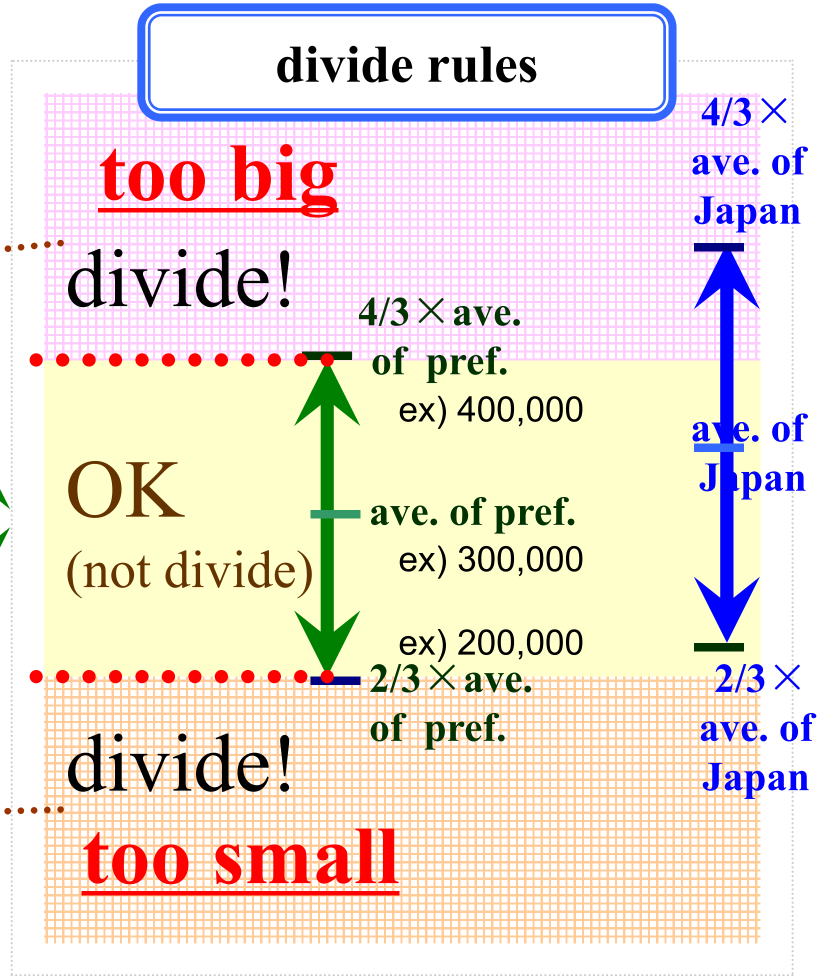
But How?
↓
gerrymander

ex) ±5%
±0%

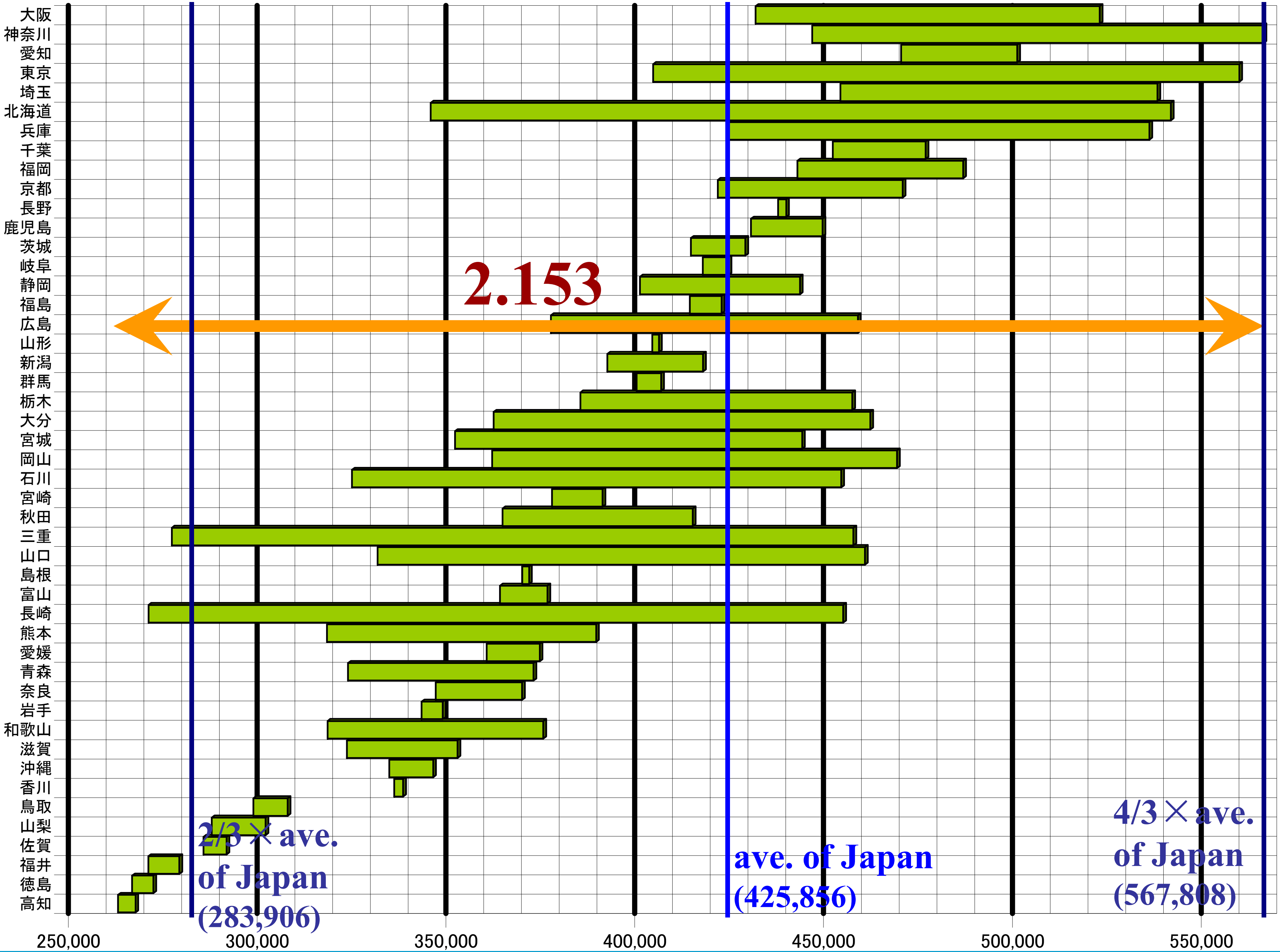
- population of a district is **too small** (< 2/3 × average)



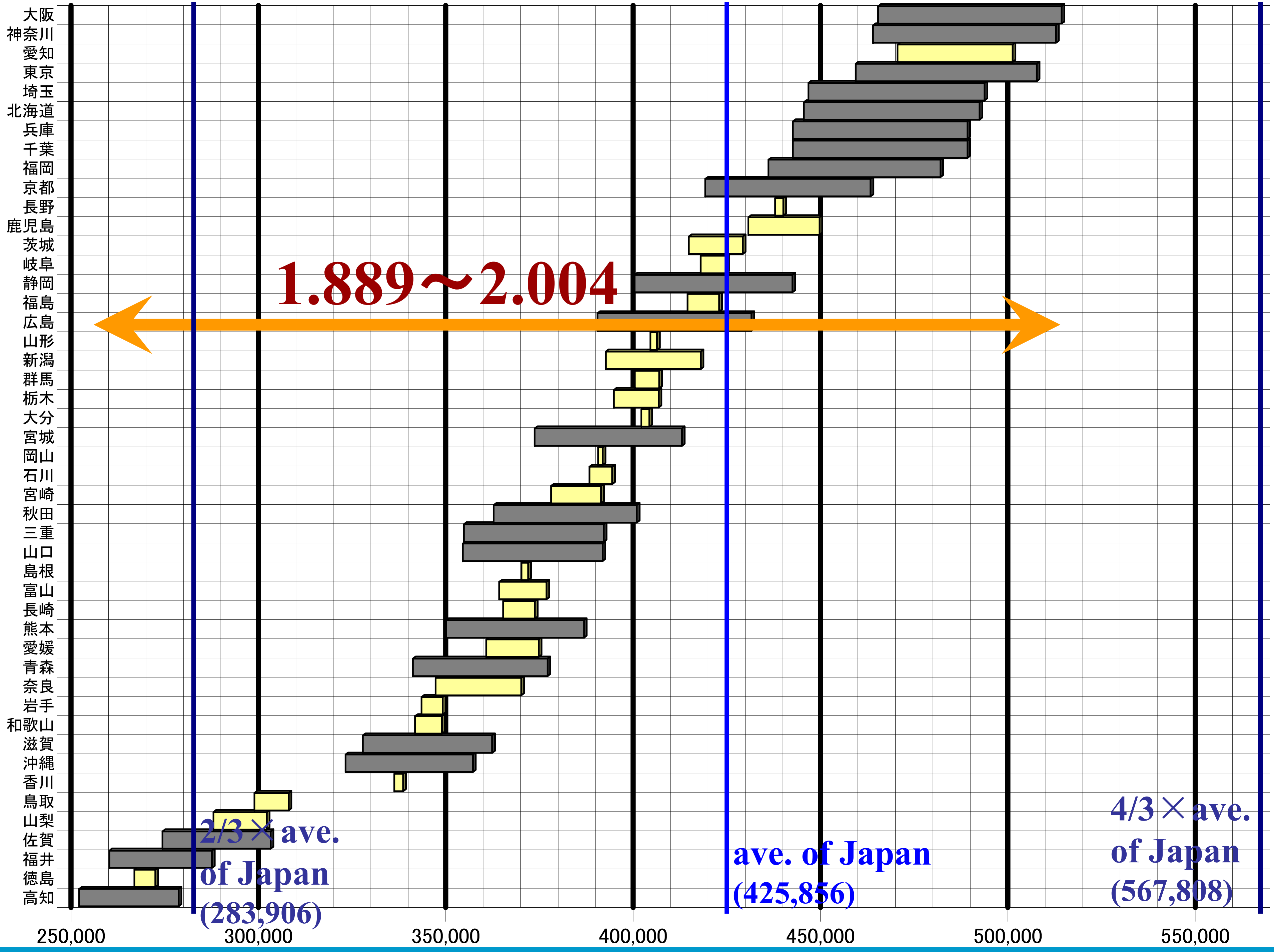
must divide somewhere



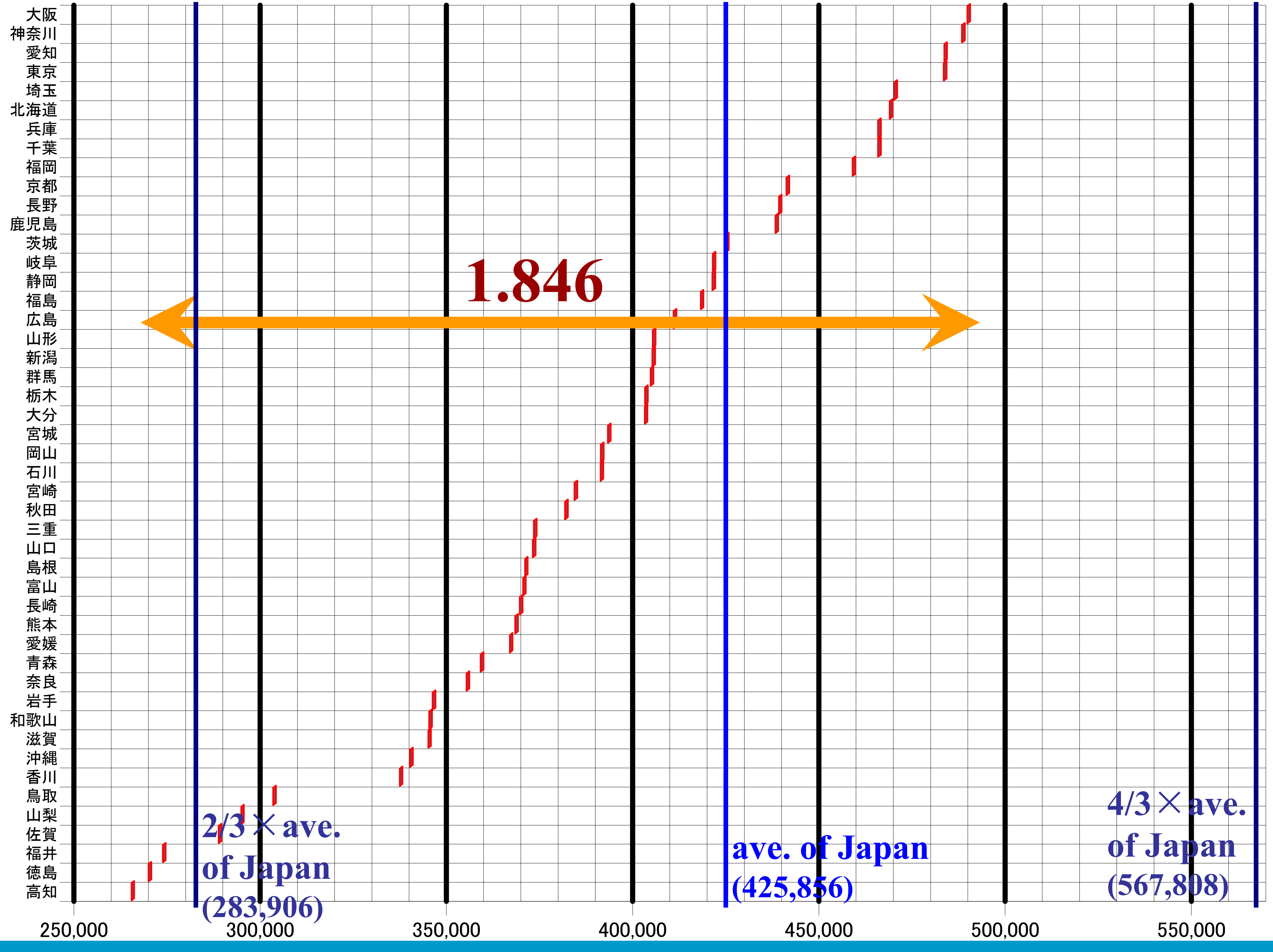
Optimal Districts [Japan type]



Optimal Districts [±5% divide rule]



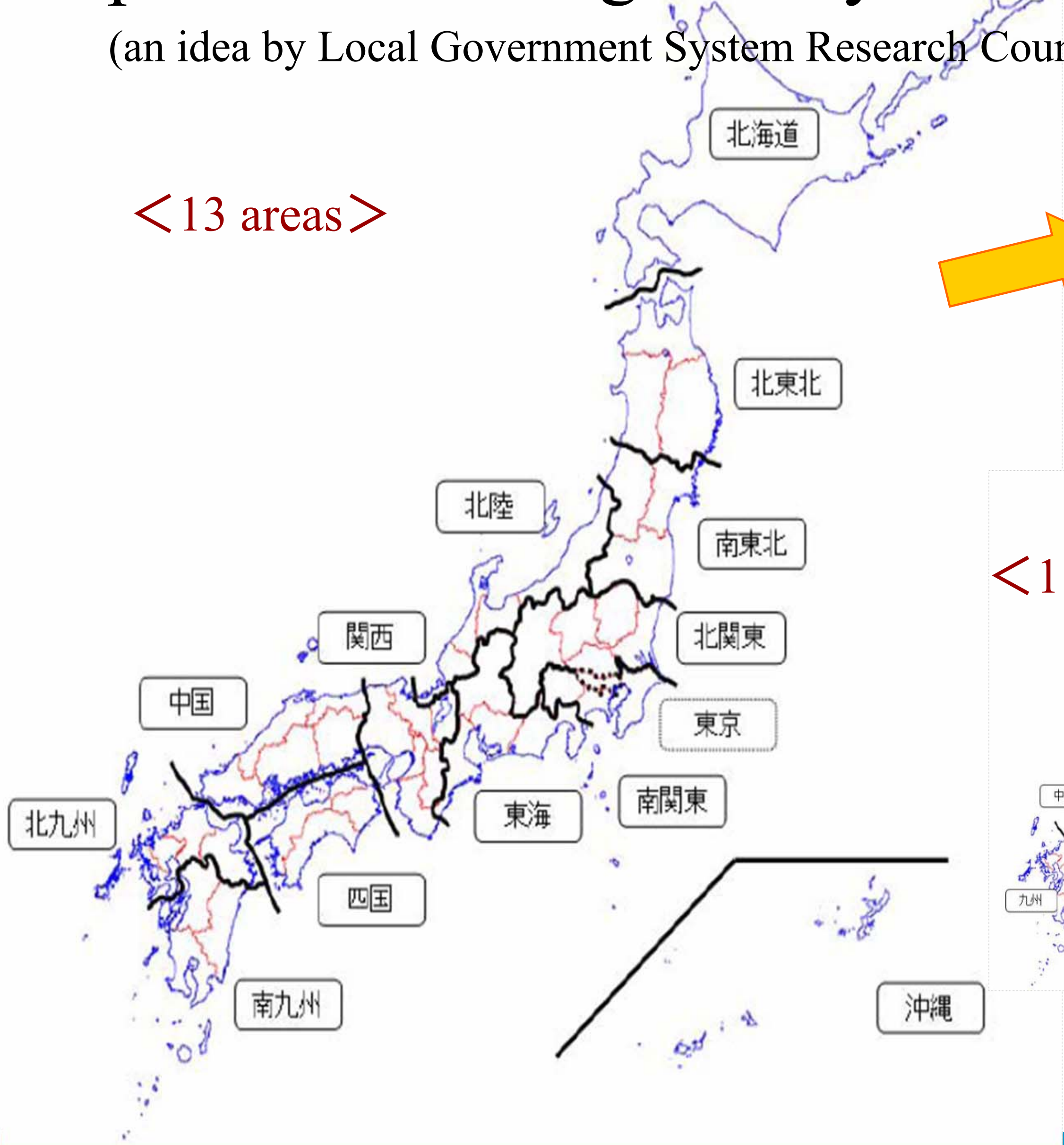
Optimal Districts [American type]



Proposal

- prefecture → regional system
  (an idea by Local Government System Research Council)

<13 areas>

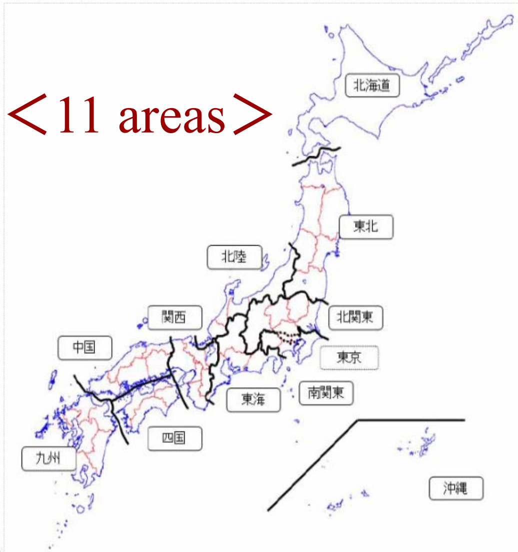


But, <u>too many</u> to solve!

area name	population	seat
1 Hokkaido	5,627,424	13.21
2 Tohoku	9,634,466	22.62
3 N.Kanto/Shinetsu	11,642,927	27.34
4 M.Kanto	35,356,183	83.02
5 Chubu	17,306,944	40.64
6 Kansai	21,714,274	50.99
7 Chugoku/Shikoku	11,761,745	27.62
8 Kyushu	13,352,022	31.35
9 Okinawa	1,360,830	3.20

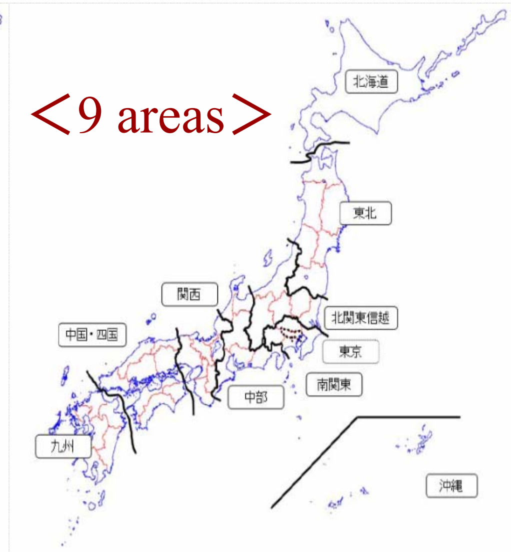
→ **<u>1.123</u>** [lower bound]

<11 areas>



→ **1.110**
[lower bound]

<9 areas>



→ **1.083**
[lower bound]

Proposal

- ultimate apportion method to minimize disparity (2004)

INPUT 300 seats pop. of 47 pref. → Apportionment Prob. by **1+LRM** → OUTPUT 47 apportioned seats → INPUT + pop. of cities → 47 Redistricting Prob. → OUTPUT 47 redistricting plan

INPUT 300 seats pop. of 47 pref. → 47 Redistricting Prob. → OUTPUT each opt. sol. → INPUT +300seats → Apportionment Prob. → OUTPUT 47 redistricting plan

several apportioned possibility

modeled as Knapsack type problem

ultimate apportion method to minimize disparity

ex) Tokyo-to

1+24.039, 1+**LD**, 1+**SD**, 1+**AMD**, 1+**GMD**, 1+**HMD**

→ 1+24=**25**, 1+25=**26**, 1+26=**27**, 1+22=**23**, 1+24=**25**, 1+24=**25**, 1+23=**24**

→ We solve the districting prob. for 23, 24, 25, 26, or 27 seats

→

	seats	opt. upper	opt. lower
Tokyo	23	574,244	499,178
Tokyo	24	540,722	446,698
Tokyo	25	536,000	421,504
Tokyo	26	536,000	394,703
Tokyo	27	536,000	376,789

limit!

1.722

(pop. census 2000)

Proposal

- Solve the Knapsack-type Problem.

$$
\begin{aligned}
\min.\ & u / l \\
s.t.\ & \sum_{j \in J} u_{ij} x_{ij} \le u \quad (i \in \{1,\ldots,47\}) \\
& \sum_{j \in J} l_{ij} x_{ij} \ge l \quad (i \in \{1,\ldots,47\}) \\
& \sum_{j \in J} x_{ij} = 1 \quad (i \in \{1,\ldots,47\}) \\
& \sum_{i \in \{1,\ldots,47\}} \sum_{j \in J} \gamma_{ij} x_{ij} = D \\
& x_{ij} \in \{0,1\} \quad (i \in \{1,\ldots,47\},\ j \in J)
\end{aligned}
$$

u_{ij}: the largest population on opt. sol. for each apportioned seat

l_{ij}: the smallest population on opt. sol. for each apportioned seat

D: the number of seats

γ_{ij}: the ideal allocation

Conclusions

1. We proposed the 300 optimal districts for the first time in Japan. The limit is **1.977**. Consequently, we offered an index of *gerrymandering*.

2. We derived the ratios for each prob. apportioned by several methods. The minimum limit is **1.750**.

3. We proposed a new framework with the Knapsack type prob. The limit is **1.750**.We also proposed a new framework with the Knapsack type prob. called the ultimate apportion method to minimize disparity. The limit is **1.722**.

4. We derived the ratios for each prob. with 280 ～ 320 members and by several apportioned methods.
   The minimum limit is **1.704**

5. We show the limit **2.153** in 2006 map.

Future works

- A main cause of the disparity is
  - ✓ Districting phase No!
  - ✓ +1 seat rule No!
  - ✓ Apportion methods No!
  - ✓ Decision process No!
  - ✓ The number of seats No!

What is a main cause?

Conjecture 1

divide rule

(exceptional provision)

Conjecture 2

Districting

for each prefecture

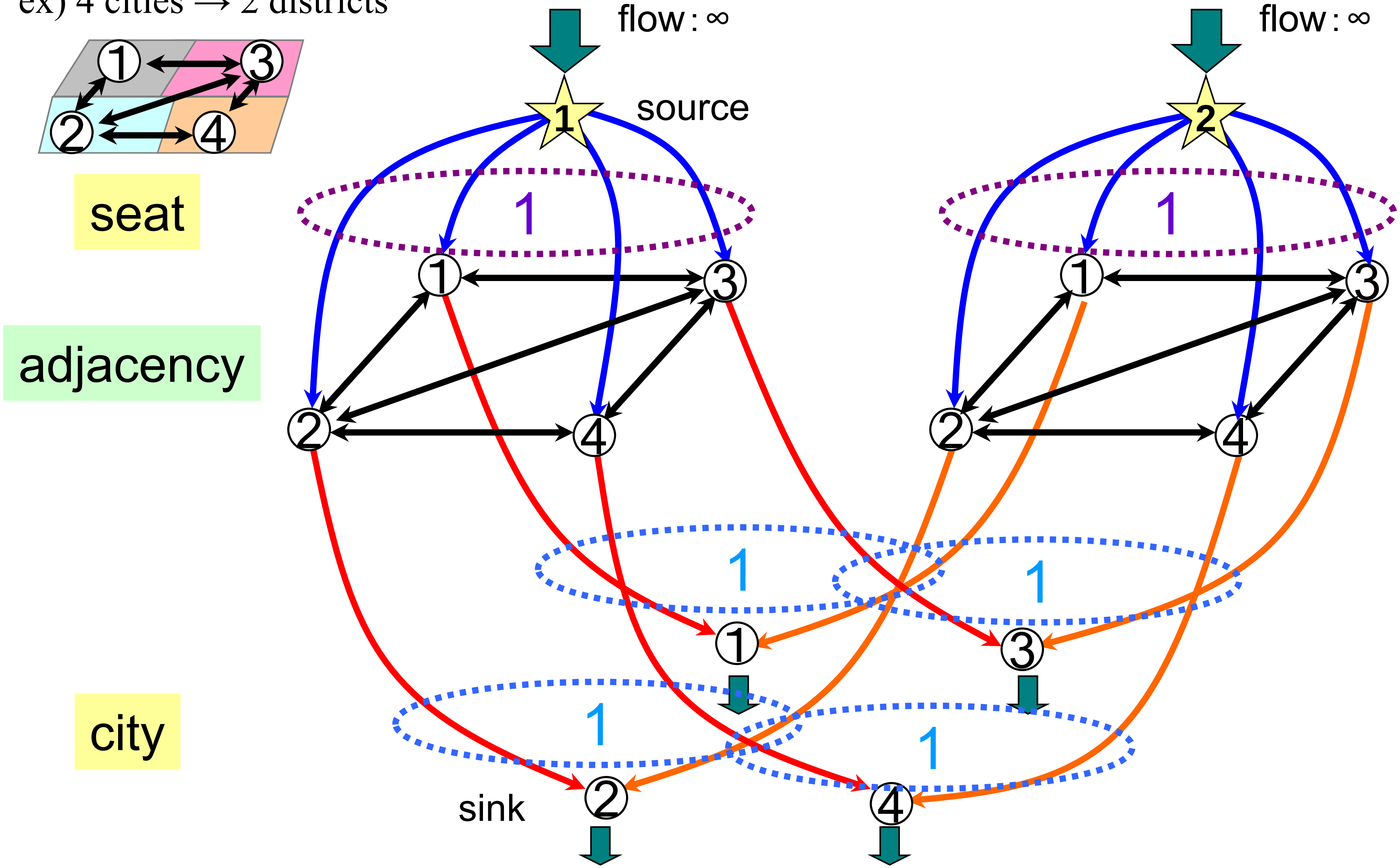
Relax prefectural boundary restriction?

faster methods for bigger problems ← further work

Thank you !

Graph Partition type

ex) 4 cities → 2 districts



Graph Partition type